Abstract

The theory of games usually concentrates attention in the market dynamical analysis on finding an the equilibrium and on the optimization behavior of sellers. It should be mentioned that for the application of the developed results. It is very important to know that the equilibrium doesn’t only exists but also can be reached by market participants. By this reason the above approach was criticized by some researchers. (For example the Austrian market process theory (Mises[21], Hayek[12] and Kirzner[18]) and the orthodox neoclassical theory (Marshall, Robinson, Chamberlin and Walras)). The problem of the equilibrium stability has been discussed in our papers earlier [7 and 8]. We have proposed to consider the problem of the attainability of the market equilibrium as an asymptotic stability task with some specially constructed iterations procedures. The result of its analysis was the theorem of the sufficient condition of the competition at the market for enterpreneurs (33). We try to renew the mentioned approach in this paper. It will be found the conditions of the local stability of the equilibrium in cases, when compete two and more products.

Definitions:

1. $n$ homogeneous products compete at the market.
2. The commercial demand of the market is $S$.
3. $a_i$ is the quality of the product $i$.
4. The entrepreneur, which produces the product $i$, use the marketing strategy $x_i \geq 0$. It generalizes all sell costs of the product $i$, somehow: advertisement, discounts, wrapping, etc. The product $i$ will disappear from the market, when $x_i = 0$.
5. The entrepreneur, who produces the product $j$, receives a profit $g_j^i$ from the sale of every piece.
6. $a_i g_i \geq a_{i+1} g_{i+1}, \forall i, i = 1 \div n - 1$
7. The sales $V_j$ and the net profit $P_j$ of the product $j$ are:
\[ V_j = S \frac{a_j x_j}{\sum_{i=1}^{n} a_i x_i}, \quad j = 1 \div n \]
\[ P_j = V_j g_j - x_j = S g_j \frac{a_j x_j}{\sum_{i=1}^{n} a_i x_i} - x_j, \quad j = 1 \div n \]

If the entrepreneur maximize the net profit from the sales of the product \( i \), we receive:
\[ \frac{\partial P_j}{\partial x_j} = \frac{S a_j g_j}{\sum_{i=1}^{n} a_i x_i} \left( 1 - \frac{a_j x_j}{\sum_{i=1}^{n} a_i x_i} \right) - 1 = 0 \text{ or } a_j x_j^* = \frac{\sum_{i=1}^{n} a_i x_i^*}{S a_j g_j} \left( S a_j g_j - \sum_{i=1}^{n} a_i x_i^* \right) \quad (1) \]

The current information of the competitors’ strategies is unknown in reality. However, every competitor receives a data about the marketing strategies of his rivals. For this reason, the market strategy for the product \( i \) depends from the previous marketing strategies of all competitors. In this case the market strategy \( x(i, m) \) is at the sale session \( m \):
\[ a_j x_j(m) = \frac{\sum_{i=1}^{n} a_i x_i(m-1)}{S a_j g_j} \left( S a_j g_j - \sum_{i=1}^{n} a_i x_i(m-1) \right) \quad (2) \]

The model of the competition for five products can be illustrated on the pictures below. There is shown the division of the market between the competitors.

![The behaviour of 5 competitors](image.png)
The evolutionary computation on the market. The competition an equilibrium in … 215

The equilibrium for 5 competitors

Fig. 2.

The pictures above describe the behavior of the five competitors with the same parameters $a_i$ and $g_i$ in both cases. Nevertheless we see at the Fig. 1. unsteady process. In the second case (Fig. 2.) the competitors move to the point of the nontrivial equilibrium. The distinction of the first case from the second one consists in the different positions of the competitors of the market. Thus the completion of the products sometimes doesn’t reach the equilibrium of the market strategies.

Now we are trying to examine the conditions of it in the general case. For the analysis of the stability around of the equilibrium is convenient to use the common application of the all marketing strategies

$$A_m = \sum_{i=1}^{n} a_i x_i(m)$$  \hspace{1cm} (3)

The region of values of the common application of all marketing strategies is always positive ($A_m > 0$), because in other case the competition not exist. Then the iteration procedure (2) transforms to the form below:

$$A_m = A_{m-1} \left( n - \frac{1}{S} \alpha_n A_{m-1} \right)$$  \hspace{1cm} (2*)

Resolving equation $A^* = A^* \left( n - \frac{1}{S} \alpha_n A^* \right)$ we have got two states of the equilibriums of the market strategies:

$$A_0^* = 0, \text{ and } A_1^* = S \frac{n-1}{\alpha_n}$$  \hspace{1cm} (4)
where \( \alpha_n = \sum_{j=1}^{n} \frac{1}{a_j g_j} \) is the computational index of the market. It characterizes the dispersion of the market parameters of the product \( i \) (the quality \( a_i \) and the profit \( g_i \)) from the average level of it. From the definition we have:

\[
\frac{n}{a_1 g_1} \leq \alpha_n \leq \frac{n}{a_n g_n}
\]

and when \( \frac{a_1 g_1}{a_n g_n} \approx 1 \), that \( \alpha_n \approx \sum_{i=1}^{n} a_i g_i \) \hspace{1cm} (5)

The competition around the equilibrium can be described by the function:

\[
y_m = y_{m-1} (2 - n) - \frac{1}{S} \alpha_n y_{m-1}^2
\]

where \( y_m = A_m - A^* \). The condition of the stability around the equilibrium is, when \( \lim_{m \to \infty} y_m = 0 \). The trivial equilibrium \( A^*_0 = 0 \) is not stable, because from the linear of equation (2*) in the vicinity of it one can write an equality \( |A_m| = |A_{m-1}| n \). The behavior of two competitors can be illustrated at the picture below:

![The competition of 2 sellers at the trivial equilibrium](image)

Fig. 3.

Analyzing the equation (6) we can write the necessary and sufficient local asymptotical stability condition of the equilibrium \( A^*_1 \) in a form of inequality \( |2 - n| < 1 \). This means that it will be stable if and only if \( n = 2 \). In this case one has an inequality \( |y_m| < |y_{m-1}| \) in the sufficiently small vicinity of equilibrium. But in our case for two competitors linear approximation has form \( |y_m| = 0 \). For this reason we have to start nonlinear analysis of the behavior of
the two competitors. From the inequality $|y_m| < |y_{m-1}|$ follows $\frac{1}{S} \alpha_n y_{m-1} < 1$. Thus for $n = 2$ the region of the equilibrium stability has form: $|y| < \frac{S}{\alpha_2}$.

Remember (3) and (6) we have $y_{m-1} = A^{m-1} - A^* m \in N$. Thus

$$\left| A^{m-1} - \frac{S}{\alpha_2} \right| < \frac{S}{\alpha_2} \text{ or } 0 < A^{m-1} < \frac{2S}{\alpha_2}, \forall m \in N$$

This means that exists the vicinity, which guarantee to reach the equilibrium of the market strategies for both competitors. Continuing to examine the equilibrium in the case $n > 2$. We have a different behavior $y_m$ for its different initial values and the quantity of competitors:

**Fig. 4.**

**Fig. 5.**
The drawings above illustrate the behavior of $y_n$ for three (Fig. 4.) and four (Fig. 5.) competitors, when the initial signification $y_0$ gives the possibility to reach the equilibrium.

**Fig. 6.**
The pattern above illustrates the behavior of $y_n$ for five competitors. Thus, we can see that possibility to reach the equilibrium depends from the initial position $y_0$ and the number of competitors $n$.

Consequently, the local stability around the equilibrium exists, when the inequity is responsible:

\[
\left| \frac{y_m}{y_{m-1}} \right| < 1 \Rightarrow \left| (2 - n) - \frac{\alpha_n}{S} y_{m-1} \right| < 1, n \in N.
\]

Thus we receive the sufficient condition of the local stability around the equilibrium:

\[
S \frac{1 - n}{\alpha_n} < y_m < S \frac{3 - n}{\alpha_n}, m \in N. \tag{7}
\]

To comparison (7) with the definition of the function $y_m$, we have received, that the region of the function exist only in the case, when $n \leq 3$. Because $y_m = A_m - A^*$ we receive two inequities, which describes the sufficient condition of the equilibrium stability for $n \leq 3$:

\[
\frac{2S}{\alpha_n} < \sum_{i=1}^{n} a_i x_j (m) < \frac{2S(n-1)}{\alpha_n}, m \in N.
\]
The evolutionary computation on the market. The competition an equilibrium in ...

From (4) we have received the inequality for the state $A^*_1$, when $n \leq 3$.

$$\frac{2S}{\alpha_n} < \frac{2S(n-1)}{\alpha_n} \left(1 - \frac{(n-1)}{\alpha_n a_i g_i} \right) < \frac{S(n-1)}{\alpha_n}$$

The right part of the inequality is always true, when $a_i g_i > \frac{n-1}{\alpha_n}$. Hence we have the condition, when the three products can to compete around the nontrivial equilibrium $A^*_1$:

$$a_i g_i > \frac{4}{\alpha_n}, i \in \{1, 2, 3\} \hspace{1cm} (8)$$

**Conclusion 1:**

The product $i$ can to compete at the market if the equation above is true for its market characteristics. The equation (6) can be used only for the analyses of behavior of 2 and 3 competitors.

Examining (1) further we propose that the marketing strategies of the competitors have used information about the marketing strategies of his rivals. Hence we can find the market strategy $x^*_i$ in the state of the equilibrium:

$$x^*_i = \frac{1}{a_i} \left( \sqrt{Sa_i g_i} - \sqrt{\sum_{j=1, j\neq i}^{n} a_j x^*_j} \right) \sqrt{\sum_{j=1, j\neq i}^{n} a_j x^*_j}, i = 1 \div n \hspace{1cm} (1^*)$$

$$\arg \max_{x_i} P(i) = \frac{1}{a(i)} \left( \sqrt{Sa_i g_i} - \sqrt{\sum_{j=1, j\neq i}^{n} a_j x^*_j} \right)^2, i = 1 \div n$$

In this case the market strategy $x_i(m)$ at the sale session $m$ have been another. As in the model above the current information of the competitors’ strategies is unknown. To suppose that $i$ competitor have used the previous marketing strategies of his rivals. In this case the market strategy $x_i(m)$ is:

$$x_i(m) = \frac{1}{a_i} \left( \sqrt{Sa_i g_i} - \sqrt{\sum_{j=1, j\neq i}^{n} a_j x_j(m-1)} \right) \sqrt{\sum_{j=1, j\neq i}^{n} a_j x_j(m-1)}, i = 1 \div n, m \in N$$

Sales of the product $i$ is unprofitable, if $P_i(m) < 0$. Hence we can receive the borders of the area of the competition:
\[
\frac{S_{a_i} g_{i} x_{i}(m)}{a_{i} x_{i}(m-1) + \sum_{j=1, j \neq i}^{n} a_{j} x_{j}(m-1)} - x_{j}(m) \geq 0, i = 1 \div n, m \in N
\]

Hence \(0 \leq \sum_{j=1, j \neq i}^{n} a_{j} x_{j}(m-1) \leq S_{a_j} g_{j}, \forall i = 1 \div n, n \in N\) and
\(0 \leq x_{i}(m) \leq S_{a_i} g_{i}, \forall i, i = 1 \div n, n \in N\).

Thus, the behavior of the all producers of the homogeneous products \(i\), where \(i = 1 \div n \& m \in N\) can be described by the new iteration procedure:

\[
x_{i}(m) = \begin{cases} 
\frac{1}{a_{i}} \left( \sqrt{S_{a_i} g_{i}} - \sqrt{\sum_{j=1, j \neq i}^{n} a_{j} x_{j}(m-1)} \right) \sqrt{\sum_{j=1, j \neq i}^{n} a_{j} x_{j}(m-1)}, & \text{when } \sum_{j=1, j \neq i}^{n} a_{j} x_{j}(m-1) \leq S_{a_i} g_{i}, i = 1 \div n, m \in N \\
0, & \text{when } \sum_{j=1, j \neq i}^{n} a_{j} x_{j}(m-1) > S_{a_i} g_{i}, i = 1 \div n, m \in N
\end{cases}
\] (9)

The set \(X^n = \{x_{j} : 0 \leq x_{j}(m) \leq S g_{j}, j = 1 \div n, m \in N\}\) is invariant relatively to the iteration procedure (9).

At the first step we examine the situation, when two competitors (the products 1 and 2) appear at the market. The behavior of both rivals can be described by next equations:

\[x_{1}(m+1) = \frac{1}{a_{1}} \left( \sqrt{S_{a_1} g_{1} a_2 x_{2}(m)} - a_{2} x_{2}(m) \right), \text{if } S_{a_1} g_{1} \geq a_{2} x_{2}(m)\]

\[x_{2}(m+1) = \frac{1}{a_{2}} \left( \sqrt{S_{a_2} g_{2} a_1 x_{1}(m)} - a_{1} x_{1}(m) \right), \text{if } S_{a_2} g_{2} \geq a_{1} x_{1}(m)\]

The competition begins, when \(x_{i}(m) > 0, i = \{1,2\}, m \in N\). These conditions are right, when the inequality \(0 \leq x_{i}(m) \leq S \frac{a_{2} g_{2}}{a_{1}}, m \in N\) is true. Hence we have received the inequality above:

\[0 \leq \sqrt{S_{a_1} g_{1} a_2 x_{2}(m-1)} - a_{2} x_{2}(m-1) \leq S_{a_2} g_{2}\]

It is right, when the inequality \(a_{1} g_{1} - 4a_{2} g_{2} \leq 0\) is true. Thus we receive the sufficient condition for the case, when two competitors always reach the equilibrium of their market strategies in the set \(X^2\). This was proved in [11].
The equilibrium of the market strategies for the two competitors can be found from the system of equations:

\[
\begin{aligned}
& a_1 x_1 = \sqrt{S a_1 x_1 a_2 x_2} - a_2 x_2 \\
& a_2 x_2 = \sqrt{S a_2 x_2 a_1 x_1} - a_1 x_1 \\
\end{aligned}
\]

Thus we have found two states of equilibrium for each of two competitors:

\[
B_0 = (0; 0) \quad \text{and} \quad B_1 = \left( S g_1 \frac{1}{(1 + \frac{a_2 g_2}{a_1 g_1})^2}; S g_2 \frac{a_2 g_2}{a_1 g_1} \frac{1}{(1 + \frac{a_2 g_2}{a_1 g_1})^2} \right)
\]

From the definition (6) we receive that

\[
\frac{a_2 g_2}{a_1 g_1} < 1.
\]

By this reason: \( x_{1, B_i} < S g_1 \) and \( x_{1, B_i} < S g_2 \). Thus \( B_0 \in X^2, B_1 \in X^2 \). These results can be illustrated by the drawing below:

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**Fig. 7.**
The picture shows us the behavior of the two competitors, when (10) is true. If both competitors use the strategies from the set $X^2$ their strategies reach the equilibrium in $B_i$ after a few sessions $m$ at any case.

The new procedure permits to define the region of local asymptotical stability. The necessary and sufficient condition of the local equilibrium stability in vicinity of the state $B_i$ depends on the eigenvalues of Yacoby matrix for the iteration procedure (9). This eigenvalues should be situated inside of the circle of the unit radius with the centre in 0. For the case $n = 2$ we have received:

$$\frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2} \frac{\partial f_2}{\partial x_1} \leq 1, \quad \frac{\partial f_1}{\partial x_1} = \frac{\partial f_2}{\partial x_2} = 0.$$

Since the definition (6), $\frac{\partial f_2}{\partial x_2} = \frac{a_1 g_1 - a_2 g_2}{2a_1 g_2}$ and $\frac{\partial f_2}{\partial x_1} = \frac{a_2 g_2 - a_1 g_1}{2a_2 g_2}$, we have the next inequity:

$$\frac{(a(1)g(1) - a(2)g(2))^2}{4a(1)g(1)a(2)g(2)} \leq 1$$

That we have found the necessary and sufficient condition of the local stability in the state $B_i$:

$$1 \leq \frac{a(1)g(1)}{a(2)g(2)} \leq 3 + 2\sqrt{2} \quad (11)$$

These results can be illustrated by the pictures below:

![The consumer 1 disloges the 2nd competitor.](image)

**Fig. 8.**
The pictures above illustrate the influence of the origin positions of the competitors at the market with their relation of the market characteristics responded inequality $4 \leq \frac{a(1)g(1)}{a(2)g(2)} \leq 3 + 2\sqrt{2}$. The Fig. 8. shows us, what can happen the second competitor chooses his market strategies very far from equilibrium. In the second case, we have concluded that there exists the bounded attractor if the initial position of two competitors.

Continue the investigation of the equilibrium in a general case. The equilibrium of the market strategies for all homogeneous products on the market $S$ means, that $x_i(m+1) = x_i(m), \forall i, m, i = 1 \div n, m \in N$. To resolve the system of the equations (1), we have received two solutions:

$$B_0^* : x_i^* = 0, i = 1 \div n,$$
$$B_1^* : x_i^* = \frac{S(n-1)}{a_i} \sum_{k=1}^{n} \frac{1}{a_k g_k} - \frac{n-1}{a_i g_i}, i = 1 \div n \quad (4*)$$

The trivial equilibrium $B_0^* = 0$ isn’t stable, because it situated out of the region of the values $x_i > 0$.

If we use the summary competition index $\alpha_i$ of the market, we receive the new form of $B_1^*$:

$$x_i^* = \frac{S(n-1)}{a_i} \alpha_i \frac{n-1}{a_i g_i} = \frac{\alpha_n}{\alpha_i^2}$$
The profit in the nontrivial equilibrium $B^*_i : x^*_i, i = 1 \div n$ is:

$$P^*_i = Sg_i (1 - \frac{n-1}{\sum_{k=1}^{n} a_i g_i})^2 = Sg_i (\frac{\alpha_i}{\alpha_n^2} - \frac{n-1}{a_i g_i})^2$$

If we summarize $a_i x^*_i$ at the equilibrium $B^*_i$ by all $i, i = 1 \div n$, we receive:

$$\sum_{i=1}^{n} a_i x^*_i = \sum_{i=1}^{n} S(n-1) (\frac{\alpha_i}{\alpha_n^2} - \frac{n-1}{a_i g_i}) = S(n-1) (n\alpha_n - (n-1)\sum_{i=1}^{n} \frac{1}{a_i g_i}) = \frac{S(n-1)}{\alpha_n} = A^*_i$$

Thus we prove, that both (2 and 9) iteration procedures have the same nontrivial equilibrium. Hence we can use the results above and the procedure $y_m$ to the further analysis of the competition. The both iteration procedures show us the definitive role of the computational index of the market $\alpha_n = \sum_{j=1}^{n} \frac{1}{a_j g_j}$. It reflects the dispersion of the market parameters of the products (the quality $a_i$ and the profit $g_i$). As is seen above, the correlation of the quality $a_i$ and the profit $g_i$ of product $i$ with this index determinate its part of the market. This gives the possibility for the entrepreneur to make a decision of entering the market and to avoid the risk to lose money with the production of noncompetitive commodities.

II. The equilibrium stability of the competition in quantities.

The concept of competitive equilibrium underlies the modern economic theory. According to known "to theorems of well-being ", competitive equilibrium is an optimum condition of market economy. Consequence of its deviation is decrease in economic efficiency. The description of conditions of the perfect competition is not constructive, because:

- it does not allow to define for the concrete market, whether these conditions are satisfied;
- how much can deviate the prices from equilibrium by Walras.

Above we have studied a competition of several firms in sale costs at the one-product market, which is close to A.Cournout’s model. The question of stability
of equilibrium competition in quantities is studied based on generalization of imperfect competition. We expand a model, having made following assumptions:

1. In the market compete the firms making similar (homogeneous) products with quality \( a_i \), where \( 0 < a_i \leq 1, \forall i, i = 1, \ldots, n \). There is \( j, 1 \leq j \leq n \), at which \( a_j = 1 \) (the standard of quality);

2. Each firm has an opportunity to make production equal to the maximal size of demand \( S_0 \). During a production cycle (trade session), the firm \( i \) delivers the certain volume of production on the market \( q_i \). It cannot changed during session. Each firm takes the quantity set by its competitors as a given, evaluates its residual demand, and then behaves as a monopoly;

3. Function of demand depends form \( Q = \varphi(S_0, \gamma, p) \), where \( \gamma \) is elasticity of demand by the price \( p \). All the products during session are on sale under one price \( p \). The market price is set at a level such that demand equals the total quantity produced by all firms. The price is established depending on total quantity of the goods, which has been supplied during the session on the market by all firms;

4. At an output during session on the market firm \( i \) spends sale costs \( x_i \). It generalizes a marketing strategy of the \( i \) product, somehow: advertisement, discounts, wrapping, etc. The sell costs and quality of the product has determined the part of market, which \( i \) firm can control during a session. During the sale session, the firms compete in sell costs, and choose their sell costs simultaneously. The product \( i \) disappears from the market, if \( x_i \leq 0 \);

5. Each firm has a cost function. Normally the cost functions are treated as common knowledge. The cost functions are the same among firms. The cost function of firm \( i \) consists of marginal costs \( c_i \geq 0 \), if \( i < j \), that \( c_i \leq c_j \). (proportional to volume of made production), and sell costs \( x_i \);

6. The firms are economically rational and act strategically, usually seeking to maximize profit given their competitors' decisions. There is strategic behavior by all firms;

7. Firms do not cooperate.
The firm, which produces the product \( j \), receives a profit \( g_j = p - c_j \) from the sale of every piece of product. In [1] we accounted the net profit \( P_j \) from the product \( j \). If the firm \( i \) during sale session maximizes the net profit from the sales of the product \( i \). Examining (2) further we propose that the marketing strategies of the competitors have used information about the marketing strategies of his rivals. In [1] we found the market strategy \( x_i^* \) in the state of the equilibrium (3).

Hence, we shall create the general model of the market division. The share of the market of the firm will be defined as:

\[
q_i = \sum_{k=1}^{n} q_k \sum_{j=1}^{n} \frac{a_j x_j}{a_j} = Q \left( 1 - \frac{n-1}{\beta a_i g_i} \right), \quad \text{where} \quad Q = \sum_{i=1}^{n} q_i.
\]  

(9)

According to (1)-(3) profit of the firm for session is:

\[
P_i(p) = Q(p)g_i(p) \left( 1 - \frac{n-1}{\alpha_i g_i(p) \beta} \right)^2
\]

Each firm aspires to receive the maximal profit. Assuming that they iterative change quantity of the products one may use the formula \( q_i^m = q_i^{m-1} + \Delta \frac{\partial \varphi^{-1}}{\partial q_i} (m-1) \). We can show that the maximum of function of profit is reached in an internal point of a set of definition. The Fig. 1 illustrates this.

Let us study a situation for \( n = 2 \). under assumption

\[
Q = \varphi(S_0, \gamma, p) = S_0 e^{-\gamma p}
\]

From [1] we have the profit functions for every firm:

\[
P_i(p) = \varphi(S_0, \gamma, p) \frac{a_i^2 g_i^3(p)}{(a_1 g_1(p) + a_2 g_2(p))^2}
\]

Each competitor wishes to maximize own profit by changing price \( p \). If we take a derivative by \( p \) of the above and equate it to zero, we receive system of the equations:

\[
\frac{\partial p}{\partial \varphi} = \left( e^{\gamma p} \frac{g_j}{(a g_j + a g_j)^2} \right) = -p e^{\gamma p} \frac{g_j}{(a g_j + a g_j)^2} + \frac{3g_j (a g_j + a g_j)^2}{(a g_j + a g_j)^2} e^{\gamma p} = 0, j = 1, 2
\]
The evolutionary computation on the market. The competition an equilibrium in ... 227

\[(3 - \gamma g_i)(a_1 g_1 + a_2 g_2) - 2(a_i + a_2)g_i = 0, j = 1, 2\]

(10)

Then the set of points of an extremum is defined by equality \(g_2 - g_1 = 0\) and \(a_1 g_1 + a_2 g_2 = 0\).

The later leads to equality \(g_1 = g_2 = 0\). If \(c_1 \neq c_2\), it is not true. From \(g_2 - g_1 = 0\) and (6) we receive

\[(3 - \gamma g_1)(a_1 g_1 + a_2 g_2) - 2(a_i + a_2)g_1 = 0\]

We have found coordinates of a condition of equilibrium on plane \(\{g_1, g_2\}\):

\[-\gamma \hat{g}_1 + 1 = 0, -\gamma \hat{g}_2 + 1 = 0\]

Hence, at an output on the market with the prices of both firms reach an extremum with the prices: \(\hat{p}_1 = c_1 + \frac{1}{\gamma}, \hat{p}_2 = c_2 + \frac{1}{\gamma}\)

To assume, that both firms try to maximize their profit, changing at \(m\) at occurrence on the market size \(g(m)\) by recurrent procedure:

\[
\begin{align*}
g_1(m) &= g_1(m-1) + \Delta \left[ (3 - \gamma g_1(m-1))(a_1 g_1(m-1) + a_2 g_2(m-1)) - g_1(m-1)2(a_i + a_2) \right] \\
g_2(m) &= g_2(m-1) + \Delta \left[ (3 - \gamma g_2(m-1))(a_1 g_1(m-1) + a_2 g_2(m-1)) - g_2(m-1)2(a_i + a_2) \right]
\end{align*}
\]

The Fig. 2 illustrate this behaviour.

**Fig. 10. A profit function**
Fig. 11. Behaviour of two competitors with the different margin costs and different initial conditions

We can show, that the extremum $\hat{p}_1 = c_1 + \frac{1}{\gamma}, \hat{p}_2 = c_2 + \frac{1}{\gamma}$ found above is a unique stable state of equilibrium, because the Jacoby matrix described above iterative procedure has negative own values.

$$J = \begin{pmatrix} -a_i - 3a_2 & 2a_2 \\ 2a_i & -3a_i - a_2 \end{pmatrix},$$

$$\text{Det}\{J\} = (a_i + 3a_2)(3a_i + a_2) - 4a_i a_2 = 3a_i^2 + 3a_2^2 + 6a_i a_2 = 3(a_i + a_2)^2 > 0,$$

$$\text{Sp}\{J\} = -4a_i - 4a_2 < 0$$

The analysis above has shown us, that strategy of optimization of quantity conducts to the different prices for each firm. It also means that the optimum size of the supply for each of firms will be distinguished.

$$Q_i = S_0 e^{-\gamma s_i - 1}, i = 1, 2$$

**Conclusion 2:**

The supply of firm does not depend on quality of the products if it adheres to optimum marketing strategy.

Then we can present profit function of $i$ firm as

$$P_i(q) = Qg_i \left(1 - \frac{n - 1}{\alpha_i g_i \beta} \right)^2 = p \frac{q_i^2}{Q} \quad (11)$$

For further analysis, we have substituted the function of demand as

$$Q = \frac{S_0}{1 + \gamma p} \quad (12)$$

At non-significant numbers of $\gamma$ this function is indistinguishable from (12). Besides it, unlike the standard linear approximation of function of demand, this function keeps property of "camber". It is extremely important for the analysis of
profit function. From here we receive, that \( p = \frac{1}{\gamma} \left( \frac{S_0}{Q} - 1 \right) \). Hence the profit function receives a form:

\[
P_i(q) = \frac{q_i^2}{Q} \left( \frac{1}{\gamma} \left( \frac{S_0}{Q} - 1 \right) - c_i \right) = \frac{S_0q_i^2}{\gamma Q^2} - \frac{q_i^2}{Q} \left( \frac{1}{\gamma} + c_i \right)
\]

As function of profit can accept any values, we shall find its regular maximum.

\[
\frac{\partial P_i}{\partial q_i} = \frac{S_0q_i}{\gamma Q^2} \left( 2 - \frac{2q_i}{Q} \right) - q_i \left( \frac{1}{\gamma} + c_i \right) \left( 2 - \frac{q_i}{Q} \right)
\]

From here we shall receive system of the linear equations from which it is possible to find the price of the equilibrium:

\[
\frac{S_0}{Q} (Q - q_i) - (1 + \gamma c_i)(2Q - q_i) = 0 \quad or \quad q_i = 2Q \cdot \frac{S_0 - Q - c_i \gamma Q}{2S_0 - Q - c_i \gamma Q}
\]

Optimizing behaviour of the five firms, we receive a following pattern. Thus, the quota of the weakest competitors gradually decreases. The Fig. 3 illustrates, that after a while there is stabilization of the total offer and, as consequence, the prices and shows us, that the quota of the weakest competitors gradually decreases. Summarizes both parts (10), we receive the equation from which is possible to find an equilibrium condition of the market:

\[
\sum_{i=1}^{n} \frac{S_0 - Q - c_i \gamma Q}{2S_0 - Q - c_i \gamma Q} = n - S_0 \sum_{i=1}^{n} \frac{1}{2S_0 - Q - c_i \gamma Q} = \frac{1}{2}
\]

From here it is possible to find market equilibrium in quantities:

\[
\sum_{i=1}^{n} \frac{1}{2S_0 - (1 + \gamma c_i)Q} = \frac{2n - 1}{2S_0}, \forall i, i = 1, n
\]

or price:

\[
\sum_{i=1}^{n} \frac{1 + \gamma p}{1 + 2\gamma p - \gamma c_i} = \frac{2n - 1}{2}, \sum_{i=1}^{n} \frac{1 - \gamma c_i}{1 + 2\gamma p - \gamma c_i} = n - 1, i = 1, n
\]

**Conclusion 3:**

Necessary condition of competitiveness of the product at market is performance of an inequality \( c_i < \frac{1}{\gamma} \), i.e. the margin cost of product should not exceed value the return of elasticity of demand by price.

If all firms have identical margin costs it is possible to find their quotas. They are equal \( q_i = q_i = q_n, \forall i, i = 1, n \). Then:
\[ Q = nq = 2S_0 \frac{n - 1}{(2n - 1)(1 + c\gamma)} \]

The division of the market under the above conditions is characterized by Fig. 4

**Fig. 12. Division of the market between five market between five firms at different entry conditions**

**Fig. 13. Division of the market between five market between five equal competitors and same initial positions**

**Conclusion 4.**

The greater number of contestant firms with the identical margin cost in the market leads to increase in the offer and reduction of price. The price in state of equilibrium, exceeding cost price, since an inequality (11) is always true, when \( n > 1 \).

\[
p = \frac{1}{\gamma} \left( \frac{S_0}{Q} - 1 \right) = \frac{1}{\gamma} \left( \frac{(2n - 1)(1 + c\gamma)}{2(n - 1)} - 1 \right) > c
\]

Let us consider a case of two firms (duopoly), when \( c_1 < c_2 \). Profit of the manufacturer for session:

\[
P_i(q) = \frac{S_0q_i^2}{\gamma Q^2} - \frac{q_i^2}{Q} \left( \frac{1}{\gamma} + c_i \right), \text{ where } Q = q_1 + q_2
\]
The evolutionary computation on the market. The competition an equilibrium in ...

Its maximum is reached when:
\[ \frac{\partial P}{\partial q_i} = S_0 q_i \left( 1 - \frac{2q_i}{Q} \right) - q_i \left( 1 + c_i \right) \left( 1 - \frac{q_i}{Q} \right) = q_i \left( \frac{2S_0}{Q} (Q - q_i) - (1 + \gamma c_i)(2Q - q_i) \right) = 0 \]

or \[ \frac{2S_0}{Q} (Q - q_i) - (1 + \gamma c_i)(2Q - q_i) = 0 \]

Hence, we receive the system of the equations defining behaviour of competitors:
\[
\begin{align*}
2 \frac{S_0}{Q} q_2 - (1 + \gamma c_1)(Q + q_2) &= 0 \\
2 \frac{S_0}{Q} q_1 - (1 + \gamma c_2)(Q + q_1) &= 0
\end{align*}
\]

(15)

Fig. 5 illustrates us, that both firms, maximising profit, after several outputs on the market reach equilibrium.

Fig. 14. Division of the market in duopoly with different margin costs and initial positions

Fig. 15. Behaviour of two firms with the same margin costs and different initial positions
Let's designate \( b_1 = 1 + \gamma c_1 \) and \( b_2 = 1 + \gamma c_2 \). From (12) it is received, accordingly:

\[
q_2 = Q \frac{b_1}{2S_0 - Q} = Q^2 \frac{b_1}{2S_0 - Qb_1} \quad \text{and} \quad q_1 = Q \frac{1 + b_2}{2S_0 - b_2Q} = Q^2 \frac{1 + b_2}{2S_0 - b_2Q} \quad (16)
\]

\[
q_1 + q_2 = Q^2 \left( \frac{b_2}{2S_0 - b_2Q} + \frac{b_1}{2S_0 - b_1Q} \right) \quad \text{or} \quad 2S_0(b_2q_2 - b_1q_1) - b_1b_2(q_2^2 - q_1^2) = 0
\]

It is a hyperbole on a plane, passing through the beginning of coordinates. The second equation \( q_1 + q_2 = 2S_0 \) is straight line. For these reasons, the solution must have not more, then two roots.

Thus, ehey correspond to two states of the market equilibrium:

\[
Q_{1,2} = \frac{2S_0}{3b_1b_2} \left( b_1 + b_2 \pm \sqrt{b_1^2 - b_1b_2 + b_2^2} \right) \quad (17)
\]

From (13) and (14) we receive two states of equilibrium: \( E_1 = (\bar{q}_1, \bar{q}_2) \) and \( E_2 = (\bar{q}_1, \bar{q}_2) \). In a state \( E_1 = (\bar{q}_1, \bar{q}_2) \) the quotas of the both firms are:

\[
\bar{q}_1 = \frac{2S_0}{3b_1b_2} \frac{b_1b_2 - (b_1 + b_2)\sqrt{b_1^2 - b_1b_2 + b_2^2}}{2b_1 - b_2 + \sqrt{b_1^2 - b_1b_2 + b_2^2}}, \quad \bar{q}_2 = \frac{2S_0}{3b_1b_2} \frac{b_1b_2 - (b_1 + b_2)\sqrt{b_1^2 - b_1b_2 + b_2^2}}{2b_2 - b_1 + \sqrt{b_1^2 - b_1b_2 + b_2^2}}
\]

Similarly, we can recalculate quotas for the state \( E_2 = (\bar{q}_1, \bar{q}_2) \):

\[
\bar{q}_1 = \frac{2S_0}{3b_1b_2} \frac{b_1b_2 - (b_1 + b_2)\sqrt{b_1^2 - b_1b_2 + b_2^2}}{2b_1 - b_2 + \sqrt{b_1^2 - b_1b_2 + b_2^2}}, \quad \bar{q}_2 = \frac{2S_0}{3b_1b_2} \frac{b_1^2 + b_2 + \sqrt{b_1^2 - b_1b_2 + b_2^2}^2}{3b_1b_2 - 3b_2 - b_1 + \sqrt{b_1^2 - b_1b_2 + b_2^2}}
\]

From here we shall find the prices for both states of equilibrium \( E_1 = (\bar{q}_1, \bar{q}_2) \) and \( E_2 = (\bar{q}_1, \bar{q}_2) \).
The evolutionary computation on the market. The competition an equilibrium in … 233

\[ \bar{p} = \frac{1}{\gamma} \left( \frac{3b_2}{2(b_1 + b_2 + \sqrt{b_1^2 - b_1b_2 + b_2^2})} - 1 \right) \]

If condition \( p > c_2 \) is true, the competition can exist. From here it is possible to find a necessary condition of the equilibrium existence. We can show, that

1) for the state \( E_1 = (\bar{q}_1, \bar{q}_2) \) inequality

\[ \frac{3b_1b_2}{2(b_1 + b_2 - \sqrt{b_1^2 - b_1b_2 + b_2^2})} > b_2 \]

is always true

2) for a state \( E_2 = (\bar{q}_1, \bar{q}_2) \) inequality

\[ \frac{3b_1b_2}{2(b_1 + b_2 + \sqrt{b_1^2 - b_1b_2 + b_2^2})} < b_2 \]

is always true. Hence, the second state of equilibrium does not satisfy to conditions of a competition.

Now we shall define conditions at which the equilibrium price will be stable. We shall consider iterative procedure from (12). Let each of competitors to aspire to maximize the profit, it during each subsequent moment increases (or takes away) the offer on some size. As, it to aspire to reach the maximal effect, it tries to come nearer to a maximum. Proceeding from it, it connects each subsequent \( m-th \) step \( \Delta \) with value of derivative function \( \frac{\partial P_i}{\partial q_i} \) have arrived in a point \( m-1 \). It means that its offer should aspire to the decision of the equation. It can be reached changing quantity of its own offer \( q_i(m) \) under the formula:

\[ q_i^m - q_i^{m-1} = \frac{\Delta}{\gamma} (2S_0q_2^{m-1} - b_1(q_1^{m-1} + 2q_2^{m-1})(q_1^{m-1} + q_2^{m-1})) \]

Fig. 7 illustrates how two firms without dependence from entry conditions of a competition come to a point of balance.

Let's analyse its stability. For the analysis of stability, it is had following functions:

\[ f(q_1, q_2) = 2S_0q_2 - b_1(q_1 + 2q_2)(q_1 + q_2), \quad \frac{\partial f}{\partial q_1} = -b_1(2q_1 + 3q_2), \quad \frac{\partial f}{\partial q_2} = 2S_0 - b_1(3q_1 + 4q_2) \]
\[ h(q_1, q_2) = 2S_0 q_1 - b_2 (2q_1 + q_2)(q_1 + q_2), \quad \frac{\partial h}{\partial q_1} = 2S_0' - b_2 (3q_2 + 4q_1), \quad \frac{\partial h}{\partial q_2} = -b_2 (3q_1 + 2q_2) \]

That the condition of equilibrium would be stable the Jacoby matrix should have a positive determinant and the negative sum of the main diagonal elements in a state \( E_{1,2} \).

We receive, that the state of equilibrium \( E_1 = (\bar{q}_1', \bar{q}_2') \) is stable always, but \( E_2 = (\bar{q}_1, \bar{q}_2) \) - unstable. These inequalities are fair at any \( b_1, b_2 \). Fig. 8 illustrates us the vice-versa situation, when \( b_1 > b_2 \). In this case, we receive a pattern similar Fig. 7

![Fig. 16. Behaviour of two firms depending initial positions in the market](image)

![Fig. 17. Behaviour of two firms in vice-versa on case](image)

In case \( c_1 = c_2 = c \) we receive, system of the equations:

\[
\begin{align*}
2 \frac{S_0}{Q} q_2 - (1 + \gamma c)(Q + q_2) &= 0 \\
2 \frac{S_0}{Q} q_1 - (1 + \gamma c)(Q + q_1) &= 0
\end{align*}
\]
The evolutionary computation on the market. The competition an equilibrium in … 235

The division of the market in this case characterizes Fig. 15. From (10) we receive, that

\[ q_1 = q_2 = q = \frac{S_0}{3(1 + \gamma c)}, Q_1 = 2q = 0, Q_2 = 2q = \frac{2S_0}{3(1 + \gamma c)} \]

Condition of stability in the state \( Q_2 \) is always carried out:

\[
J = \begin{pmatrix}
-\frac{5}{3}S_0 & -\frac{1}{3}S_0 \\
\frac{1}{3}S_0 & -\frac{5}{3}S_0
\end{pmatrix}, \quad \text{Det}\{J\} = \frac{8}{3}S_0^2 > 0, \quad \text{Sp}\{J\} = -\frac{10}{3}S_0 < 0
\]

Hence, the equilibrium price for two competitors with the equal margin costs will be:

\[ p = \frac{1}{\gamma} \left( \frac{S_0}{Q} - 1 \right) = \frac{1}{2\gamma} + 1.5c \quad \text{(18)} \]

Conclusion 5:

The competitive price of the product in the state of equilibrium is defined by an inequality \( p < \frac{2}{\gamma} \), i.e. the price in the state of equilibrium is less than size to a return half of elasticity of demand.

The Fig. 18 illustrates how two competitors with the identical cost price can reach an equilibrium.

Fig. 18. An equilibrium in case of duopoly with the equal margin costs

Conclusion 6.

There is one stable state of equilibrium. Hence, the price, which has established in the market, exceeds the cost price of worst of competitors.
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